**Fall 2008-2009** **Quiz 2** **Date: December 1**
**Prof: M. Egeileh** **Math 201 - Sections 17 to 20** **Duration: 1 hour**

---

**Exercise 1** *(2 pts)* Write the Fourier series generated by the function $f : [0, 2\pi] \longrightarrow \mathbb{R}$ defined by : $f(x) = 27\cos(10x) - 35\sin(100x)$ (*Advice : do not spend more than 1 minute on this exercise*).

**Exercise 2** *(48 pts)*

1. In the plane $\mathbb{R}^2$, we are interested in the curve $\mathcal{P}$ of polar equation $r = \dfrac{1}{1+\cos\theta}$.
   (a) By considerations of periodicity and symmetry, find an optimal interval $I$ for $\theta$ that will enable you at the end of this part to draw the entire curve $\mathcal{P}$. *(5 pts)*
   (b) Find, *when they exist*, the value(s) of $\theta$ in $I$ for which the curve $\mathcal{P}$ : i) intersects with the $x$-axis, ii) intersects with the $y$-axis, iii) passes by the pole $O$. Then, indicate precisely the tangent lines to $\mathcal{P}$ at the points corresponding to these values of $\theta$. *(7 pts)*
   (c) If $M$ is the point of $\mathcal{P}$ corresponding to the angle $\theta$, what can you say about the distance from the pole to $M$ when $\theta \to \pi$ ? *(5 pts)*
   (d) Using the results you obtained in the preceding questions, sketch the curve $\mathcal{P}$. Do you recognize the geometrical nature of $\mathcal{P}$ ? Check that your guess is correct by writing a cartesian equation for $\mathcal{P}$. *(6 pts)*

2. Let $f$ be the function defined by : $f(x,y) = \dfrac{\frac{1}{2} - x}{y^2 - 1 + 2x}$.
   (a) Find and draw quickly the domain of definition $D$ of $f$. *(5 pts)*
   (b) Is $D$ an open subset of $\mathbb{R}^2$ ? a closed subset of $\mathbb{R}^2$ ? Both open and closed ? Neither open nor closed ? Is it bounded or unbounded ? Justify all your answers. *(10 pts)*
   (c) Does $\displaystyle\lim_{(x,y)\to(\frac{1}{2},0)} f(x,y)$ exist ? If yes, compute it. If no, prove that it does not exist. *(10 pts)*

**Exercise 3** *(50 pts)* Let $\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the function defined by $\sigma(\theta,\varphi) = (x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi))$, where :

$$\begin{cases} x(\theta,\varphi) &= \frac{1}{3}(\sin\theta)(\cos\varphi) \\ y(\theta,\varphi) &= (\sin\theta)(\sin\varphi) \\ z(\theta,\varphi) &= \frac{1}{2}\cos\theta \end{cases}$$

We also consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $f(x,y,z) = 9x^2 + y^2 + 4z^2$, and we set $h = f \circ \sigma$, that is, $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the function defined by :

$$h(\theta,\varphi) = f\left(\frac{1}{3}(\sin\theta)(\cos\varphi)\ ,\ (\sin\theta)(\sin\varphi)\ ,\ \frac{1}{2}\cos\theta\right)$$

1. (a) By using chain rule (without evaluating $h(\theta,\varphi)$), compute $\dfrac{\partial h}{\partial\theta}(\theta,\varphi)$ and $\dfrac{\partial h}{\partial\varphi}(\theta,\varphi)$. *(10 pts)*
   (b) Give a cartesian equation for the level surface $\mathcal{E}$ of $f$ passing by the point $(0,-1,0)$. *(4 pts)*
   (c) Show that for any $(\theta,\varphi) \in \mathbb{R}^2$, the point $\sigma(\theta,\varphi)$ belongs to $\mathcal{E}$. *(5 pts)*

(d) Can you explain the results obtained in question (a) ? *(6 pts)*

2. (a) Write the gradient of $f$ at a point $(x, y, z)$ of $\mathbb{R}^3$. Deduce the unit vector giving the direction in which $f(x, y, z)$ will decrease most rapidly, starting from the point $(0, -1, 0)$. *(6 pts)*

   (b) Give a cartesian equation for the tangent plane $P$ to $\mathcal{E}$ at $(0, -1, 0)$. *(4 pts)*

   (c) What is the geometrical nature of $\mathcal{E}$ ? Sketch the shape of $\mathcal{E}$ in $\mathbb{R}^3$. Indicate on the same figure the vector $\vec{\nabla} f(0, -1, 0)$, as well as the tangent plane $P$. *(5 pts)*

3. Starting from the point $(\pi, 2\pi)$ in the $\theta\varphi$-plane $\mathbb{R}^2$, the value $g(\theta, \varphi)$ of a differentiable function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ increases most rapidly in the direction of the vector $\vec{u} = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$. Find the value of $6\,\frac{\partial g}{\partial \theta}(\pi, 2\pi) - 2\,\frac{\partial g}{\partial \varphi}(\pi, 2\pi)$. *(10 pts)*

4. (a) What is the topological boundary (set of all boundary points) of $\mathcal{E}$ ? *(3 extra pts)*

   (b) We can define a different notion of boundary, the *geometrical boundary* : a "two-dimensional ant", walking on a CD (Compact Disc), getting away from the center, reaches at some point the *geometrical boundary* of the CD. What can you say about the *geometrical boundary* of $\mathcal{E}$ ? (*Hint : Let the ant walk on $\mathcal{E}$...*) *(1 extra pt)*